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About summability of Fourier-Laplace series

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Abstract

In this paper we study the almost everywhere convergence of the expansions related to the self-adjoint extension of the Laplace operator. The sufficient conditions for summability is obtained. For the orders of Riesz means, which greater than critical index $\frac{N-1}{2}$ we established the estimation for maximal operator of the Riesz means. Note that when order α of Riesz means is less than critical index then for establish of the almost everywhere convergence requests to use other methods form proving negative results. We have constructed different method of summability of Laplace series, which based on spectral expansions property of self-adjoint Laplace-Beltrami operator on the unit sphere.

Key words: spectral expansion, spectral functions, Riesz means, almost everywhere convergence.

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1. INTRODUCTION

Let S^N be unit sphere in R^{N+1} . Let us denote by $\lambda_0, \lambda_1, \dots$ the distinct eigenvalues of the Laplace operator $-\Delta_S$, arranged in increasing order. Let H_k denote the eigenspace corresponding to λ_k . We call elements of H_k spherical harmonics of degree k . It is well known (see [14]) that $\dim H_k = a_k$:

$$a_k = \begin{cases} 1, & \text{if } k = 0, \\ N, & \text{if } k = 1, \\ \frac{(N+k)!}{N!k!} - \frac{(N+k-2)!}{N!(k-2)!}, & \text{if } k \geq 2 \end{cases} \quad (1.1)$$

Let \hat{A} is a self-adjoint extension of the Laplace operator Δ_S in $L_2(S^N)$ and if E_λ is the corresponding spectral resolution, then for all functions $f \in L_2(S^N)$ we have

$$\hat{A}f = \int_0^\infty \lambda dE_\lambda f.$$

The operator \hat{A} has in $L_2(S^N)$ a complete orthonormal system of eigenfunctions

$$\{Y_1^{(k)}(x), Y_2^{(k)}(x), \dots, Y_{a_k}^{(k)}(x)\} \subset H_k, k = 0, 1, 2, \dots,$$

corresponding to the eigenvalues $\{\lambda_k = k(k + N - 1)\}, k = 0, 1, 2, \dots$

It is easy to check that the operators E_λ have the form

$$E_\lambda f(x) = \sum_{\lambda_n < \lambda} Y_n(f, x), \quad (1.2)$$

where

$$Y_k(f, x) = \sum_{j=1}^{a_k} Y_j^{(k)}(x) \int_{S^N} f(y) Y_j^{(k)}(y) d\sigma(y) \quad (1.3)$$

The Riesz means of the partial sums (1.2) is defined by

$$E_n^\alpha f(x) = \sum_{k=0}^n \left(1 - \frac{\lambda_k}{\lambda_n}\right)^\alpha Y_k(f, x) \quad (1.4)$$

The most convenient object for a detailed investigation are the expansions of the form (1.4). The integral (1.4) may be transformed writing instead of \hat{A} the integral to the right in (1.3) and then changing the order of integration. This yields the formula

$$E_\lambda^s f(x) = \int_{S^N} \Theta^s(x, y, \lambda) f(y) d\sigma(y) \quad (1.5)$$

with

$$\Theta^\alpha(x, y, n) = \sum_{k=0}^n \left(1 - \frac{\lambda_k}{\lambda_n}\right)^\alpha Z_k(x, y). \quad (1.6)$$

For $\alpha = 0$ this kernel is called the spectral function of the Laplace operator for the entire space S^N .

The behavior of the spectral expansion corresponding to the the Laplace-Beltrami operator is closely connected with the asymptotical behavior of the kernel $\Theta^\alpha(x, y, n)$.

In the study of questions of a.e. convergence it is convenient to introduce the maximal operator

$$E_*^\alpha f(x) = \sup_{\lambda \geq 0} |E_\lambda^\alpha f(x)|.$$

The basic results of this paper is

Theorem 1.1. *Let $f \in L_p(S^N)$, $1 \leq p \leq 2$ then Riesz means of order $s > (N-1) \left(\frac{1}{p} - \frac{1}{2}\right)$ of the Fourier-Laplace series of the function f converges almost everywhere on S^N to the f .*

2. PRELIMINARIES

In this section we are going to prove estimates for maximal operator. Let us recall some more general definition of harmonic analysis.

For any two points x and y from S^N we shall denote by $\gamma(x, y)$ spherical distance between these two points. Actually, $\gamma(x, y)$ is a measure of angle between x and y . It is obvious, that $\gamma(x, y) \leq \pi$.

Spherical ball $B(x, r)$ of radius r and with the center at point x defined by $B(x, r) = \{y \in S^N : \gamma(x, y) < r\}$. For integrable function $f(x)$ the maximal function of Hardy-Littlewood

$$f^*(x) = \sup_{r>0} \frac{1}{|B(x, r)|} \int_{S^N} |f(y)| d\sigma(y) \quad (2.1)$$

is finite almost everywhere on sphere. The maximal function f^* plays a major role in analysis and has been much studied (see.[14]). In particular, for any $p > 1$ and if $f \in L_p$, then there exists constant c_p , such that

$$\|f^*\|_{L_p} \leq \frac{c_p(N)}{p-1} \|f\|_{L_p},$$

where c_p has no singularities at point $p = 1$.

Theorem 2.1. *Let $\alpha > \frac{N-1}{2}$ then for all $f \in L_1(S^N)$ we have*

$$E_*^\alpha f(x) \leq \frac{c_\alpha(N)}{\alpha - \frac{N-1}{2}} (f^*(x) + f^*(\bar{x})) \quad (2.2)$$

where $\gamma(x, \bar{x}) = \pi$.

The prove can be found in [11].

From the boundness of the maximal function we obtain

Theorem 2.2. *Let $\alpha > \frac{N-1}{2}$ then for all $f \in L_p(S^N), p > 1$ we have*

$$\|E_*^\alpha f(x)\|_p \leq \frac{c_\alpha(N)}{\alpha - \frac{N-1}{2}} \|f\|_p \quad (2.3)$$

The statement of the Theorem 2.3 we will be using when p approaches 1. For using the Stein's interpolation theorem we have to set the analogous estimation on the case $p = 2$.

Let us for $f \in L_2(S^N)$ and $\alpha > -1/2$ denote an operator M^α , which plays main role in the estimate of the Riesz means

$$M^\alpha f(x) = \sup_{n \geq 1} \left(\frac{1}{n} \sum_{k=0}^n |E_k^\alpha f(x)| \right). \quad (2.4)$$

If we consider the Riesz means of order $\alpha + \beta$ then we may connect this means with the operator M^β as follow:

Lemma 2.3. *Let $\alpha > -1/2, \beta > 1/2$ then for all $f \in L_2(S^N)$ we have*

$$E_*^{\alpha+\beta} f(x) \leq c_{\alpha, \beta} M^\beta f(x) \quad (2.5)$$

Proof. Let $\alpha > -1/2$, $\beta > 1/2$ and $f \in L_2(S^N)$, then

$$E_n^{\alpha+\beta} f(x) = \sum_{k=0}^n \left(1 - \frac{\lambda_k}{\lambda_n}\right)^{\alpha+\beta} Y_k(f, x). \quad (2.6)$$

Using analogous formula of integration by parts to partial sums we have

$$E_n^{\alpha+\beta} f(x) = \sum_{k=0}^n \left(\left(1 - \frac{\lambda_k}{\lambda_n}\right)^\beta + \left(1 - \frac{\lambda_k}{\lambda_n}\right)^\beta \right) E_k^\alpha f(x). \quad (2.7)$$

Applying the Cauchy's inequality we have

$$|E_n^{\alpha+\beta} f(x)| \leq \left(\sum_{k=0}^n | \left(1 - \frac{\lambda_k}{\lambda_n}\right)^\beta - \left(1 - \frac{\lambda_k}{\lambda_n}\right)^\beta |^2 \right)^{1/2} \left(\sum_{k=0}^n |E_k^\alpha f(x)|^2 \right)^{1/2}. \quad (2.8)$$

In view that

$$\left(n \sum_{k=0}^n | \left(1 - \frac{\lambda_k}{\lambda_n}\right)^\beta - \left(1 - \frac{\lambda_{k+1}}{\lambda_n}\right)^\beta |^2 \right)^{1/2} \leq \frac{1}{2} B(2\beta - 1, \frac{3}{2}), \quad (2.9)$$

where $B(x, y)$ is Beta function:

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt. \quad (2.10)$$

we have

$$E_*^{s+\alpha} f(x) \leq C_{s,\alpha} M^\alpha f(x) \quad (2.11)$$

So it is not hard to see that, if we obtain estimate for M^α , then the same estimate true for $E_*^{s+\alpha}$. For estimate M^α let enter a new function G^α , which defined as follow:

$$G^\alpha f(x) = \left(\sum_{n=1}^{\infty} \frac{1}{n} |E_n^{\alpha+1} f(x) - E_n^\alpha f(x)|^2 \right)^{1/2}.$$

for all $f \in L_2(S^N)$.

Lemma 2.4. *Let $\alpha > -1/2$, then for all $f \in L_2(S^N)$ we have*

$$\|G^\alpha(f)\|_{L_2(S^N)} \leq \text{const } \|f\|_{L_2(S^N)} \quad (2.12)$$

Proof. Using orthonormality of the functions $\{Y_k(x)\}$ and Fubini's theorem about the changeable of integration order, we may estimate the norm of the function G^α as follow

$$\|G^\alpha(f)\|_{L_2(S^N)} \leq \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^n \left(1 - \frac{\lambda_k}{\lambda_n}\right)^{2\alpha} \frac{\lambda_k^2}{\lambda_n^2} |Y_k(f, x)|^2.$$

Therefore, in view

$$\frac{1}{n} \sum_{k=1}^n \left(1 - \frac{\lambda_k}{\lambda_n}\right)^{2\alpha} \frac{\lambda_k^2}{\lambda_n^2} \leq \frac{1}{2} B(2\alpha + 1, 5/2),$$

we have

$$\|G^\alpha(f)\|_{L_2(S^N)} \leq \|f\|_{L_2(S^N)}.$$

Lemma 2.4 proved.

Lemma 2.5. *Let $\alpha > -1/2, m = 1, 2, \dots$ then for all $f \in L_2(S^N)$ we have*

$$M^\alpha(f) \leq M^{\alpha+m} + G^\alpha(f) + G^{\alpha+1}(f) + \dots + G^{\alpha+m-1}(f) \quad (2.13)$$

Proof. We prove inequality (2.13) by the Induction method. Let first $m = 1$, then we have to prove inequality:

$$M^\alpha(f) \leq M^{\alpha+1} + G^\alpha(f) \quad (2.14)$$

For proving the inequality (2.14) the form of G^α estimate form below as follow:

$$\begin{aligned} [G^\alpha f(x)]^2 &= \sum_{n=1}^{\infty} \frac{1}{n} |E_n^{\alpha+1} f(x) - E_n^\alpha f(x)|^2 \geq \sum_{k=1}^n \frac{1}{k} |E_k^{\alpha+1} f(x) - E_k^\alpha f(x)|^2 \geq \\ &\geq \frac{1}{n} \sum_{k=1}^n |E_k^{\alpha+1} f(x) - E_k^\alpha f(x)|^2 \geq \frac{1}{n} \sum_{k=1}^n ||E_k^{\alpha+1} f(x)| - |E_k^\alpha f(x)||^2 = \\ &= \frac{1}{n} \sum_{k=1}^n |E_k^{\alpha+1} f(x)|^2 + \frac{1}{n} \sum_{k=1}^n |E_k^\alpha f(x)|^2 - 2 \frac{1}{n} \sum_{k=1}^n |E_k^{\alpha+1} f(x)| |E_k^\alpha f(x)| \end{aligned}$$

So using Caushy's inequality

$$\sum |a_k| |b_k| \leq \sqrt{\sum |a_k|^2} \sqrt{\sum |b_k|^2}$$

we get

$$\begin{aligned} &\frac{1}{n} \sum_{k=1}^n |E_k^{\alpha+1} f(x)|^2 + \frac{1}{n} \sum_{k=1}^n |E_k^\alpha f(x)|^2 - 2 \frac{1}{n} \sum_{k=1}^n |E_k^{\alpha+1} f(x)| |E_k^\alpha f(x)| \geq \\ &\geq \frac{1}{n} \sum_{k=1}^n |E_k^{\alpha+1} f(x)|^2 + \frac{1}{n} \sum_{k=1}^n |E_k^\alpha f(x)|^2 - 2 \frac{1}{n} \sqrt{\sum_{k=1}^n |E_k^{\alpha+1} f(x)|^2} \sqrt{\sum_{k=1}^n |E_k^\alpha f(x)|^2} = \\ &= \left(\sqrt{\frac{1}{n} \sum_{k=1}^n |E_k^\alpha f(x)|^2} - \sqrt{\frac{1}{n} \sum_{k=1}^n |E_k^{\alpha+1} f(x)|^2} \right)^2. \end{aligned}$$

So finally for G^α we have

$$[G^\alpha(f)]^2 \geq \left(\sqrt{\frac{1}{n} \sum_{k=1}^n |E_k^\alpha f(x)|^2} - \sqrt{\frac{1}{n} \sum_{k=1}^n |E_k^{\alpha+1} f(x)|^2} \right)^2.$$

Then in view of definition of M^α we get inequality (2.14). So inequality (2.5) proved when $m = 1$. Assume that (2.5) true for all $k < m$:

$$M^\alpha(f) \leq M^{\alpha+k} + G^\alpha(f) + G^{\alpha+1}(f) + \dots + G^{\alpha+k-1}(f)$$

Now we have to extend this inequality for $k + 1$. So, as

$$M^{\alpha+k} f(x) \leq M^{\alpha+k+1} + G^{\alpha+k}$$

we have

$$\begin{aligned} & M^{\alpha+k} + G^\alpha(f) + G^{\alpha+1}(f) + \dots + G^{\alpha+k-1}(f) \leq \\ & \leq M^{\alpha+k+1} + G^\alpha(f) + G^{\alpha+1}(f) + \dots + G^{\alpha+k-1}(f) + G^{\alpha+k}(f). \end{aligned}$$

Which proves the assertion of Lemma 2.5.

Lemma 2.6. *Let $\alpha > -1/2$, then for all $f \in L_2(S^N)$ we have*

$$\|M^\alpha(f)\|_{L_2(S^N)} \leq c_\alpha \|f\|_{L_2(S^N)}. \quad (2.15)$$

Proof. If we choose the integer m in (2.5) such that, $m > \frac{N}{2}$, then we have $\alpha + m > (N - 1)/2$. It is not hard to show that for all $\alpha + m > -1/2$

$$M^{\alpha+m} f(x) \leq E_*^{\alpha+m} f(x).$$

Due to Theorem 2.3 we get

$$\|M^{\alpha+m}(f)\|_2 \leq \|f\|_2.$$

Then in view Lemma 2.4 we have

$$\|M^\alpha(f)\|_{L_2(S^N)} \leq \|M^{\alpha+m}(f)\|_{L_2(S^N)} + \|G^\alpha(f)\|_{L_2(S^N)} + \dots + \|G^{\alpha+m-1}(f)\|_{L_2(S^N)}$$

So we have

$$\|M^{\alpha+m}(f)\|_{L_2(S^N)} \leq \|f\|_{L_2(S^N)}.$$

Lemma 2.15 proved.

Finally for Riesz means we have more important

Theorem 2.7. *Let $f \in L_2(S^N)$, then for Riesz means of positive order $\alpha > 0$ we have*

$$\|E_*^\alpha(f)\|_{L_2(S^N)} \leq c_\alpha \|f\|_{L_2(S^N)}. \quad (2.16)$$

Consequently, for every $f \in L_2(S^N)$ the Riesz means $E_n^\alpha f$ of any positive order converge almost everywhere on S^N . For multiple Fourier series this result is due to Mitchell [9], for spectral expansions of the elliptic operator to Peetre [10]. The question of almost everywhere convergence of $E_n^\alpha f$, $f \in L_2(S^N)$, $N > 2$, remains open for $\alpha = 0$.

For multiple Fourier series this result due to Mitchell (1955). The question of a.e. convergence of $E_\lambda^\alpha f(x)$, $f \in L_2(S^N)$, remains open for $\alpha = 0$.

3. INTERPOLATION BETWEEN $L_p, p > 1$ AND L_2

In this section we are going to prove the almost everywhere convergence of spectral expansions by Riesz means at index under the critical line $\alpha = (N - 1) \left(\frac{1}{p} - \frac{1}{2} \right)$, $1 \leq p \leq 2$. Let us now pass to the case $1 < p < 2$. It follows from (2.3) that for such values of p the Riesz means converge above the critical order. This good result for p near 1, but its gets cruder when p approaches 2, as for $p = 2$ a.e. convergence holds true for all $\alpha > 0$. There arises the natural desire to interpolate between (2.3) and (2.16).

It is intuitively clear that for intermediate $p, 1 < p < 2$, an analogous estimate must hold with an α that decreases from $\frac{N-1}{2}$ to 0. Ordinary interpolation allows one to interpolate the inequalities (2.3) and (2.16) for fixed operator and it does not provide the possibility to change the order α when passing from L_1 to L_2 . However, if dependence of the operators under consideration on the parameter α is analytic then one can carry out the interpolation in α .

Let us state Stein's interpolation theorem in a form suitable for our purposes.

We say that a function $\phi(\tau), \tau \in R$, has admissible growth if there exist constants $a < \pi$ and $b > 0$ such that

$$|\phi(z)| \leq \exp(b \exp(a|\tau|)). \quad (3.1)$$

Let A_z be a family of operators defined for simple functions (i.e. functions which are finite linear combinations of characteristic functions of measurable subsets of S^N .) We term the family A_z admissible if for any two simple functions f and g the function

$$\phi(z) = \int_{S^N} f(x) A_z g(x) dx$$

is analytic in the strip $0 \leq Re z \leq 1$ and has admissible growth in $Im z$, uniformly in $Re z$ (this means that we have an estimate in $Im z$ which is analogous to (3.1), with constants a and b independent of $Re z$).

Theorem 3.1. *Let A_z be an admissible family of linear operators such that*

$$\|A_{i\tau}\|_{L_{p_0}(S^N)} \leq M_0(\tau) \|f\|_{L_{p_0}(S^N)}, \quad 1 \leq p_0 \leq \infty,$$

$$\|A_{1+i\tau}\|_{L_{p_1}(S^N)} \leq M_1(\tau) \|f\|_{L_{p_1}(S^N)}, \quad 1 \leq p_1 \leq \infty,$$

for all simple functions f and with $M_j(\tau)$ independent of τ and admissible growth. Then there exists for each $t, 0 \leq t \leq 1$, a constant M_t such that for every simple function f holds

$$\|A_t\|_{L_{p_t}(S^N)} \leq M_t(\tau) \|f\|_{L_{p_t}(S^N)}, \quad \frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}.$$

Some of the most useful objects to which this theorem can be applied are the Riesz means, which analytically depend on s . In this case, admissible growth in practise does not cause any great difficulty, as practically all functions encountered in the applications have exponential growth. the restriction of the domain of definition of A_z to simple functions does not diminish the possibility of interpolation, as the simple functions constitute a dense subset in $L_p(S^N)$.

Let us turn attention to the difficulties which arise then. The interpolation theorem 3.1, which we would like to apply, pertains to an analytic family of linear operators, but the maximal operator E_*^α is nonlinear. This difficulty is resolved in the following standard way. Denote by Ψ the class of positive measurable functions on S^N taking finitely many different values. If $\mu \in \Psi$ then by the definition of the maximal operator we have

$$|E_{\mu(x)}^\alpha f(x)| \leq E_*^\alpha f(x). \quad (3.2)$$

It is clear that one can pick a sequence $\mu_1 \leq \mu_2 \leq \dots$ of elements in Ψ such that

$$\lim_{k \rightarrow \infty} |E_{\mu(x)}^\alpha f(x)| = E_*^\alpha f(x).$$

This allows us to invert (3.2) as follows:

$$\sup_{\mu \in \Psi} \|E_{\mu(x)}^\alpha f(x)\|_{L_p(S^N)} = \|E_*^\alpha f(x)\|_{L_p(S^N)}. \quad (3.3)$$

Fix now $\mu \in \Psi$ and consider the linear operator E_*^α , which depends analytically on the parameter α .

Let us fix an arbitrary $\varepsilon > 0$ and set $\alpha(z) = \frac{N-1}{2}z + \varepsilon$. Then the operators $E_{\mu(x)}^{\alpha(z)}$ satisfy all conditions of interpolation theorem of Stein with $p_1 > 1$ and $p_0 = 2$

$$\|E_{\mu(x)}^{\alpha(1+i\tau)}(f)\|_{L_{p_1}(S^N)} \leq A_1 \|f\|_{L_{p_1}(S^N)}, \quad (3.4)$$

$$A_1 = \frac{Ce^{\pi|\tau|/2}}{(p_1 - 1)^2}, \quad \text{Re } \alpha(1 + i\tau) > \frac{N-1}{2}, \quad p_1 > 1,$$

and

$$\|E_{\mu(x)}^{\alpha(i\tau)}(f)\|_{L_2(S^N)} \leq A_0 \|f\|_{L_2(S^N)}, \quad \text{Re } \alpha(i\tau) > 0. \quad (3.5)$$

Consequently for all $t : 0 < t < 1$ we have

$$\|E_{\mu(x)}^{\alpha(t)}(f)\|_{L_p(S^N)} \leq A_t \|f\|_{L_p(S^N)}. \quad (3.6)$$

where $\alpha(t) = \frac{N-1}{2}t + \varepsilon$, $\frac{1}{p} = \frac{1-t}{2} + \frac{t}{p_1}$, and excluding t , we have, $\alpha > \frac{N-1}{2} \left(\frac{1}{p} - \frac{1}{2} \right)$ and A_t :

$$A_t \leq \frac{\text{const}}{(p-1)^2}.$$

Such that we have proved following

Theorem 3.2. *Let $f(x) \in L_p(S^N)$, $p > 1$. If the order α of Riesz means $E_\lambda^\alpha f(x)$ is greater than critical index $\frac{N-1}{2}$, then for maximal operator E_*^α we have*

$$\|E_*^\alpha(f)\|_{L_p(S^N)} \leq \frac{c_{p,\alpha}}{(p-1)^2} \|f\|_{L_p(S^N)}, \quad (3.7)$$

where constant $c_{p,\alpha}$ has no singularities at $p = 1$ and $\alpha = \frac{N-1}{2}$.

The estimate (3.7) says that the means $E_\lambda^\alpha f(x)$ of any function $f \in L_p(S^N)$, $1 < p < 2$, converge a.e. for the values of α indicated. Let us note that the convergence theorem is valid also for $p = 1$. In this case, the maximal function f^* is not of strong type (1,1), so the estimate (2.3) is not true, but it is of weak type (1,1), which in view (??) is sufficient for the a.e. convergence of the means $E_\lambda^\alpha f(x)$ for $\alpha > \frac{N-1}{2}$ for any integrable function f .

4. CONCLUSION

To sum up, we may conclude that a sufficient condition for the a.e. convergence of the Riesz means $E_\lambda^\alpha f(x)$ of a function $f \in L_p(S^N)$ is that we have

$$\alpha > (N-1) \left(\frac{1}{p} - \frac{1}{2} \right), \quad 1 \leq p < 2. \quad (4.1)$$

How sharp is this condition? To elucidate this question we begin with the case of Riesz means of order $\alpha = \frac{N-1}{2}$. Condition (4.1) shows that for the a.e. convergence of Riesz means one requires an order above the critical index only if $p = 1$. That this requirement is essential follows from the following theorem:

Theorem 4.1. *There exists a function $f \in L_1(S^N)$, $N \geq 2$, such that almost everywhere on S^N*

$$\lim_{\lambda \rightarrow \infty} |E_\lambda^\alpha f(x)| = +\infty, \quad \alpha = \frac{N-1}{2}.$$

Thus, for $p = 1$ condition (4.1) is sharp. For $p > 1$ we have (see [12]):

Theorem 4.2. *If*

$$0 \leq \alpha \leq N \left(\frac{1}{p} - \frac{1}{2} \right) - \frac{1}{2}, \quad 1 \leq p \leq \frac{2N}{N+1}, \quad (4.2)$$

then there exists a function $f \in L_p(S^N)$ such that $E_\lambda^\alpha f(x)$ is divergent on a set of positive measure.

Between the conditions (4.1) and (4.2) there is a gap.

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